

HAMILTONIAN CIRCLE ACTIONS WITH FIXED POINT SET ALMOST MINIMAL

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ABSTRACT. Motivated by recent works on Hamiltonian circle actions satisfying certain minimal conditions, in this paper, we consider Hamiltonian circle actions satisfying an almost minimal condition. More precisely, we consider a compact symplectic manifold (M, ω) admitting a Hamiltonian circle action with fixed point set consisting of two connected components X and Y satisfying $\dim(X) + \dim(Y) = \dim(M)$. Under some cohomology condition, we determine the circle action, the integral cohomology rings of M , X and Y , and the total Chern classes of M , X , Y , and of the normal bundles of X and Y . The results show that these invariants are unique for symplectic manifolds, while for Kähler manifolds, the S^1 -equivariant biholomorphism and symplectomorphism type is unique.

1. INTRODUCTION

Let the circle act effectively on a compact symplectic manifold (M, ω) of dimension $2n$ with moment map $\phi: M \rightarrow \mathbb{R}$. The fixed point set M^{S^1} , which is the same as the critical set of ϕ , contains at least two connected components — the minimum and the maximum of ϕ . In the case when M^{S^1} consists of exactly two connected components, X and Y , since M is compact and symplectic, by Morse-Bott theory, we must have

$$\dim(X) + \dim(Y) + 2 \geq \dim(M).$$

In [11], we studied the case when the following *minimal* condition holds:

$$(1.1) \quad \dim(X) + \dim(Y) + 2 = \dim(M).$$

Standard examples of manifolds satisfying (1.1) are \mathbb{CP}^n , and $\tilde{G}_2(\mathbb{R}^{n+2})$, the Grassmannian of oriented 2-planes in \mathbb{R}^{n+2} with $n > 1$ odd, equipped with standard circle actions. For the case when (1.1) holds, we classified the circle action, the integral cohomology rings of M , X and Y , and the total Chern classes of M , X , Y , and of the normal bundles of X and Y . It turns out that these data on M are exactly as in the two families of standard examples. In particular, $b_{2i}(M) = 1$ and $b_{2i-1}(M) = 0$ for all $0 \leq i \leq \frac{1}{2} \dim(M)$.

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Note that for a compact symplectic manifold (M, ω) , $0 \neq [\omega]^i \in H^{2i}(M; \mathbb{R})$, so we always have $b_{2i}(M) \geq 1$ for all $0 \leq 2i \leq \dim(M)$. If $b_{2i}(M) = 1$ for some $2i$, we say that the $2i$ -th Betti number of M is *minimal*. The even Betti numbers of \mathbb{CP}^n and of $\tilde{G}_2(\mathbb{R}^{n+2})$ with $n > 1$ odd are all minimal.

Originally motivated by the classical Petrie's conjecture, Tolman studied Hamiltonian circle actions on compact 6-dimensional manifolds with minimal even Betti numbers [15]. Then a number of work [12, 11, 10, 9, 14, 3, 5] appeared about Hamiltonian circle actions on compact manifolds with minimal even Betti numbers or with fixed point set satisfying a minimal condition

$$\sum_{F \subset M^{S^1}} (\dim(F) + 2) = \dim(M) + 2,$$

where the F 's are connected components of the fixed point set. These two minimal conditions are closely related [11, Sec. 4]. Another recent work [4] studied Hamiltonian circle actions on compact manifolds with fixed point set consisting of two components.

We know an example, $\tilde{G}_2(\mathbb{R}^{2n+2})$, of dimension $4n$, equipped with a standard Hamiltonian circle action, whose fixed point set consists of two connected components X and Y satisfying the following *almost minimal* condition

$$(1.2) \quad \dim(X) + \dim(Y) = \dim(M).$$

Note that $b_{2i}(\tilde{G}_2(\mathbb{R}^{2n+2})) = 1$ for $2i \neq 2n$, $b_{2n}(\tilde{G}_2(\mathbb{R}^{2n+2})) = 2$, and $b_{2i-1}(\tilde{G}_2(\mathbb{R}^{2n+2})) = 0$ for all i . So the even Betti numbers of $\tilde{G}_2(\mathbb{R}^{2n+2})$ are almost minimal.

In this paper, we look at Hamiltonian circle actions on compact manifolds with fixed point set consisting of two connected components satisfying (1.2). Let us first look at the standard example and the data on it.

Example 1.3. Let $n \geq 1$, and let $\tilde{G}_2(\mathbb{R}^{2n+2})$ be the Grassmannian of oriented 2-planes in \mathbb{R}^{2n+2} . This $4n$ -dimensional manifold is a coadjoint orbit of $SO(2n+2)$, so it is a symplectic (Kähler) manifold and it admits a Hamiltonian $SO(2n+2)$ action.

There is a Hamiltonian $S^1 \subset SO(2n+2)$ action on $\tilde{G}_2(\mathbb{R}^{2n+2})$ induced by the S^1 action on $\mathbb{R}^{2n+2} \cong \mathbb{C}^{n+1}$ given by

$$\lambda \cdot (z_1, \dots, z_{n+1}) = (\lambda z_1, \dots, \lambda z_{n+1}).$$

The fixed point set consists of two connected components X and Y , where

$$X \cong Y \cong \mathbb{CP}^n,$$

corresponding to the two orientations on the real 2-planes in $\mathbb{P}(\mathbb{C}^{n+1})$. The $S^1/\mathbb{Z}_2 \cong S^1$ action on $\tilde{G}_2(\mathbb{R}^{2n+2})$ is *semifree* (i.e., free outside fixed points).

Let $[\omega]$ be a primitive integral class on $\tilde{G}_2(\mathbb{R}^{2n+2})$ represented by the symplectic form ω such that $[\omega|_X] = u$ and $[\omega|_Y] = v$ are also primitive integral (such a form exists by Lemma 3.1 when $n \geq 2$. If $n = 1$, $\tilde{G}_2(\mathbb{R}^{2n+2})$

is diffeomorphic to $S^2 \times S^2$, we take $\omega = x_1 + x_2$, where x_1 and x_2 are positive generators of $H^2(S^2 \times \text{pt}; \mathbb{Z})$ and $H^2(\text{pt} \times S^2; \mathbb{Z})$ respectively. Then the total Chern classes of $\tilde{G}_2(\mathbb{R}^{2n+2})$, X , Y , and of the normal bundles N_X of X and N_Y of Y are respectively as follows:

$$c(\tilde{G}_2(\mathbb{R}^{2n+2})) = \frac{(1 + [\omega])^{2n+2}}{1 + 2[\omega]}, \text{ in particular, } c_1(\tilde{G}_2(\mathbb{R}^{2n+2})) = 2n[\omega],$$

$$c(X) = (1 + u)^{n+1}, \quad c(Y) = (1 + v)^{n+1},$$

$$c(N_X) = \frac{(1 + u)^{n+1}}{1 + 2u}, \text{ and } c(N_Y) = \frac{(1 + v)^{n+1}}{1 + 2v}.$$

The integral cohomology groups of $\tilde{G}_2(\mathbb{R}^{2n+2})$ are:

$$H^i(\tilde{G}_2(\mathbb{R}^{2n+2}); \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i \text{ is even and } i \neq 2n, \\ \mathbb{Z} \oplus \mathbb{Z} & \text{if } i = 2n, \\ 0 & \text{if } i \text{ is odd.} \end{cases}$$

The integral cohomology ring of $\tilde{G}_2(\mathbb{R}^{2n+2})$ is

$$H^*(\tilde{G}_2(\mathbb{R}^{2n+2}); \mathbb{Z}) = \begin{cases} \mathbb{Z}[x, y]/(x^{n+1} - 2xy, y^2) & \text{if } n \text{ is odd,} \\ \mathbb{Z}[x, y]/(x^{n+1} - 2xy, y^2 - x^n y) & \text{if } n \text{ is even,} \end{cases}$$

where $x = [\omega]$ and $\deg(y) = 2n$.

These data also follow from the results of this paper. In the literature, the cohomology ring $H^*(\tilde{G}_2(\mathbb{R}^{2n+2}); \mathbb{Z})$ may be given differently using different generators.

In this paper, for a general compact Hamiltonian S^1 -manifold (M, ω) with fixed point set consisting of two connected components satisfying (1.2), the essential idea is we assume one or two fixed point components have *even* degree integral cohomology ring of some complex projective space (just as in Example 1.3), we show that the important data on M are exactly as in Example 1.3. These results, in particular, imply that in the Kähler case, M is unique up to S^1 -equivariant biholomorphism and symplectomorphism.

Unlike the case when (1.1) holds, where condition (1.1) itself has implications on the even degree cohomology of X and Y , for the current case when (1.2) holds, we do need a condition on the even degree cohomology of X and Y to obtain our results. See Remark 1.10.

Remark 1.4. In our main theorems, for $\dim(M) > 4$, our assumption grants that $\dim H^2(M; \mathbb{R}) = 1$, so up to scaling, we can assume the symplectic class $[\omega]$ is primitive integral. This is not a serious assumption.

Now let us state our main results. First, the first Chern class $c_1(M)$ is an important data, in particular for the Kähler case. Assuming a condition on one fixed point component whose dimension is no lower than half dimension of the manifold M , we give a result on the action and on $c_1(M)$.

Theorem 1.5. *Let (M, ω) be a compact symplectic manifold of dimension bigger than 4 admitting an effective Hamiltonian S^1 action with moment map ϕ such that the fixed point set consists of two connected components X and Y with $\dim(X) + \dim(Y) = \dim(M)$. Assume $[\omega]$ is a primitive integral class, $\dim(X) \geq \frac{1}{2} \dim(M)$, and $H^{\text{even}}(X; \mathbb{Z}) = \mathbb{Z}[u]/u^{\frac{1}{2} \dim(X)+1}$, where $u = [\omega|_X]$. Then $\dim(X) = \frac{1}{2} \dim(M)$ and the following 3 conditions are equivalent:*

- (1) *the action is semifree,*
- (2) *$|\phi(Y) - \phi(X)| = 1$, and*
- (3) *$c_1(M) = \frac{1}{2} \dim(M)[\omega]$.*

Assuming a condition on both fixed point set components, we obtain the following results.

Theorem 1.6. *Let (M, ω) be a compact symplectic manifold of dimension bigger than 4 admitting an effective Hamiltonian S^1 action such that the fixed point set consists of two connected components X and Y with $\dim(X) + \dim(Y) = \dim(M)$. Assume $[\omega]$ is a primitive integral class, $H^{\text{even}}(X; \mathbb{Z}) = \mathbb{Z}[u]/u^{\frac{1}{2} \dim(X)+1}$ and $H^{\text{even}}(Y; \mathbb{Z}) = \mathbb{Z}[v]/v^{\frac{1}{2} \dim(Y)+1}$, where $u = [\omega|_X]$ and $v = [\omega|_Y]$. Then*

- (1) *$\dim(X) = \dim(Y) = 2n$, $\dim(M) = 4n$, where $n > 1$,*
- (2) *the action is semifree,*
- (3) *$H^*(X; \mathbb{Z}) = \mathbb{Z}[u]/u^{n+1}$ and $H^*(Y; \mathbb{Z}) = \mathbb{Z}[v]/v^{n+1}$,*
- (4) *$H^*(M; \mathbb{Z}) \cong H^*(\tilde{G}_2(\mathbb{R}^{2n+2}); \mathbb{Z})$ as rings,*
- (5) *$c(M) \cong c(\tilde{G}_2(\mathbb{R}^{2n+2}))$, and*
- (6) *$c(X)$, $c(Y)$, $c(N_X)$ and $c(N_Y)$ are all the same as in Example 1.3.*

Remark 1.7. Theorems 1.5 and 1.6 do not hold for dimension 4. For example, take $(M, \omega) = (\Sigma_g \times S^2, x_1 + 2x_2)$, where Σ_g is a surface of genus $g \geq 0$, x_1 is a positive $H^2(\Sigma_g; \mathbb{Z})$ generator and x_2 is a positive $H^2(S^2; \mathbb{Z})$ generator. Let S^1 act on M by fixing Σ_g and rotating semifreely on S^2 . Then $X \cong Y \cong \Sigma_g$, and the moment map image has length 2.

But (1), (2), and the claim for $c(N_X)$ and $c(N_Y)$ in Theorem 1.6 hold for dimension 4 (see Proposition 5.1).

Assuming $H^*(M; \mathbb{Z}) \cong H^*(\tilde{G}_2(\mathbb{R}^{2n+2}); \mathbb{Z})$ as rings, we can determine all the other data for all dimensions.

Theorem 1.8. *Let (M, ω) be a compact symplectic manifold admitting an effective Hamiltonian S^1 action such that the fixed point set consists of two connected components X and Y with $\dim(X) + \dim(Y) = \dim(M)$. Assume $H^*(M; \mathbb{Z}) \cong H^*(\tilde{G}_2(\mathbb{R}^{2n+2}); \mathbb{Z})$ as rings. Then*

- (1) *$H^*(X; \mathbb{Z}) \cong H^*(Y; \mathbb{Z}) \cong H^*(\mathbb{CP}^n; \mathbb{Z})$ as rings,*
- (2) *the action is semifree,*
- (3) *$c(M) \cong c(\tilde{G}_2(\mathbb{R}^{2n+2}))$, and*

- (4) $c(X)$, $c(Y)$, $c(N_X)$ and $c(N_Y)$ are all isomorphic to those in Example 1.3.

Let (M, ω, J) be a compact Kähler manifold of complex dimension n , and assume $[\omega]$ is an integral class. Then $c_1(M) = n[\omega]$ implies that M is biholomorphic to $\tilde{G}_2(\mathbb{R}^{n+2})$ [7]. Our main results Theorems 1.5, 1.6 and 1.8 and the method in [9] allow us to use various criteria to identify the following Hamiltonian S^1 -Kähler manifold with Example 1.3 in the S^1 -equivariant complex and symplectic categories.

Theorem 1.9. *Let (M, ω, J) be a compact Kähler manifold admitting a holomorphic Hamiltonian S^1 action with moment map ϕ such that the fixed point set consists of two connected components X and Y with $\dim(X) + \dim(Y) = \dim(M)$. Assume $[\omega]$ is a (suitable) primitive integral class. Then any one of the following conditions implies that M is S^1 -equivariantly biholomorphic and symplectomorphic to $\tilde{G}_2(\mathbb{R}^{2n+2})$ as in Example 1.3, where $2n = \dim_{\mathbb{C}}(M)$.*

- (1) $c_1(M) = \dim_{\mathbb{C}}(M)[\omega]$.
- (2) $H^*(M; \mathbb{Z}) \cong H^*(\tilde{G}_2(\mathbb{R}^{2n+2}); \mathbb{Z})$ as rings.
- (3) $H^*(X; \mathbb{Z}) = \mathbb{Z}[u]/u^{\dim_{\mathbb{C}}(X)+1}$ and $H^*(Y; \mathbb{Z}) = \mathbb{Z}[v]/v^{\dim_{\mathbb{C}}(Y)+1}$, where $u = [\omega|_X]$ and $v = [\omega|_Y]$.
- (4) $\dim_{\mathbb{C}}(M) = 2n > 2$, $H^{\text{even}}(X; \mathbb{Z}) = \mathbb{Z}[u]/u^{\dim_{\mathbb{C}}(X)+1}$ and $H^{\text{even}}(Y; \mathbb{Z}) = \mathbb{Z}[v]/v^{\dim_{\mathbb{C}}(Y)+1}$.
- (5) $\dim_{\mathbb{C}}(M) = 2n > 2$, $H^{\text{even}}(X; \mathbb{Z}) = \mathbb{Z}[u]/u^{n+1}$, and the action is semifree.
- (6) $\dim_{\mathbb{C}}(M) = 2n > 2$, $H^{\text{even}}(X; \mathbb{Z}) = \mathbb{Z}[u]/u^{n+1}$, and $|\phi(Y) - \phi(X)| = 1$.

Remark 1.10. For the case when (1.1) holds, in [11, Prop. 4.2], we showed that the condition (1.1) implies that the even degree cohomology groups of X and Y are all one dimensional. This is not true for the case when (1.2) holds. Counter examples can be created in the following way. Take a \mathbb{C}^k -bundle over S^2 , where $k > 1$. Let S^1 act on the fiber \mathbb{C}^k freely and fix the base S^2 . This action is Hamiltonian with moment map $\phi = \sum_i |z_i|^2$, where the z_i 's are the complex coordinates on \mathbb{C}^k . Take a symplectic cut [8] at a regular value r of ϕ , we get a compact Hamiltonian S^1 -manifold of dimension $2k + 2$, with fixed point set consisting of two components $X = S^2$ and $Y = \phi^{-1}(r)/S^1$. The component Y is a \mathbb{CP}^{k-1} -bundle over S^2 , so its even degree cohomology groups are not all one dimensional. Hence condition (1.2) itself does not allow us to weaken the assumptions of our theorems.

The organization of the paper is as follows. In Section 2, we give some preliminary results for the next sections. In Section 3, we prove Theorem 1.5. In Section 4, we use the cohomology condition to rule out non-semifree actions. In Section 5, we determine the equivariant Euler class of the normal bundle of the fixed point set. This is an important step for the next two

sections. In Section 6, we determine the integral cohomology ring of the fixed point set. In Section 7, we obtain the total Chern classes of the fixed point set and of its normal bundle. In Section 8, we obtain the integral cohomology ring and total Chern class of the manifold M and prove Theorem 1.6. In Section 9, we prove Theorem 1.8, and in Section 10, we consider the Kähler case and prove Theorem 1.9.

2. SOME PRELIMINARIES

In this section, we state and prove some preliminary results which we will use in the next sections. In this paper, we will use equivariant cohomology techniques. We refer to [11] for a the basic material and summary of useful facts about S^1 -equivariant cohomology.

First, let us set up some notations:

- (1) $H_{S^1}^*(M; R)$ — the S^1 -equivariant cohomology of the S^1 -manifold M with coefficient ring R .
- (2) t — a generator of $H_{S^1}^2(\text{pt}; \mathbb{Z}) = H^2(\mathbb{CP}^\infty; \mathbb{Z})$.
- (3) N_X — the normal bundle of a submanifold X in a manifold.
- (4) $e^{S^1}(N_F)$ — the S^1 -equivariant Euler class of the normal bundle N_F of a fixed point set component F in an S^1 -manifold.
- (5) $c^{S^1}(N_F)$ — the S^1 -equivariant total Chern class of the normal bundle N_F of a fixed point set component F in an S^1 -manifold.
- (6) $c^{S^1}(M)$ — the S^1 -equivariant total Chern class of the S^1 -manifold M .
- (7) Γ_F — the sum of the weights of the S^1 action on the normal bundle of the fixed point set component F in an S^1 -manifold.

The following elementary fact is essential in applications.

Lemma 2.1. *Let the circle act on a compact symplectic manifold (M, ω) with moment map $\phi: M \rightarrow \mathbb{R}$, such that M^{S^1} consists of two connected components X and Y . Assume $[\omega]$ is an integral class. Then there exists $\tilde{u} \in H_{S^1}^2(M; \mathbb{Z})$ such that*

$$\tilde{u}|_X = [\omega|_X], \text{ and } \tilde{u}|_Y = [\omega|_Y] + (\phi(X) - \phi(Y))t.$$

In particular, $\phi(X) - \phi(Y) \in \mathbb{Z}$. Moreover, if $\mathbb{Z}_k \subset S^1$ fixes any point on M , then $k \mid (\phi(X) - \phi(Y))$.

Proof. For the existence of \tilde{u} , see [11, Lemma 2.7]. If \mathbb{Z}_k fixes any point, then the submanifold $M^{\mathbb{Z}_k}$ fixed by \mathbb{Z}_k contains X and Y , and the $S^1/\mathbb{Z}_k \cong S^1$ action on $M^{\mathbb{Z}_k}$ has moment map $\phi' = \frac{\phi}{k}$. Apply the first claim on $M^{\mathbb{Z}_k}$ for the $S^1/\mathbb{Z}_k \cong S^1$ action, we get $\phi'(X) - \phi'(Y) \in \mathbb{Z}$, which means $k \mid (\phi(X) - \phi(Y))$. \square

Using Lemma 2.1, we get the following result on $e^{S^1}(N_X)$, which will be an important ingredient in our proofs.

Lemma 2.2. *Let (M, ω) be a compact symplectic manifold admitting a Hamiltonian S^1 action with moment map ϕ such that M^{S^1} consists of two connected components X and Y . Assume $[\omega]$ is an integral class. Then there exists $\lambda \in H_{S^1}^*(X; \mathbb{Z})$ such that*

$$\lambda e^{S^1}(N_X) = ([\omega|_X] + t(\phi(Y) - \phi(X)))^{\frac{1}{2} \dim(Y)+1}.$$

Proof. Since $[\omega]$ is an integral class, by Lemma 2.1, there exists $\tilde{u} \in H_{S^1}^2(M; \mathbb{Z})$ such that

$$\tilde{u}|_X = [\omega|_X] \text{ and } \tilde{u}|_Y = [\omega|_Y] + t(\phi(X) - \phi(Y)).$$

So

$$(2.3) \quad (\tilde{u} + t(\phi(Y) - \phi(X)))^{\frac{1}{2} \dim(Y)+1}|_Y = 0.$$

Let $M^- = \{m \in M \mid \phi(X) < \phi(m)\}$. Consider the long exact sequence for the pair (M, M^-) in equivariant cohomology:

$$\begin{array}{ccccccc} \cdots & \rightarrow & H_{S^1}^*(M, M^-; \mathbb{Z}) & \rightarrow & H_{S^1}^*(M; \mathbb{Z}) & \rightarrow & H_{S^1}^*(M^-; \mathbb{Z}) \rightarrow \cdots \\ & & \downarrow \cong & & \downarrow & & \downarrow \cong \\ & & H_{S^1}^{*- \text{codim}(X)}(X; \mathbb{Z}) & \rightarrow & H_{S^1}^*(X; \mathbb{Z}) & \rightarrow & H_{S^1}^*(Y; \mathbb{Z}) \end{array}$$

Here, the first vertical map is the Thom isomorphism, the second vertical map is the restriction, the third vertical map is an isomorphism since M^- is homotopy equivalent to Y , and the first horizontal map on the second row is multiplication by $e^{S^1}(N_X)$. Combining (2.3), we get that there exists $\lambda \in H_{S^1}^{\dim(Y)+2-\text{codim}(X)}(X; \mathbb{Z})$ such that

$$\begin{aligned} \lambda e^{S^1}(N_X) &= (\tilde{u} + t(\phi(Y) - \phi(X)))^{\frac{1}{2} \dim(Y)+1}|_X \\ &= ([\omega|_X] + t(\phi(Y) - \phi(X)))^{\frac{1}{2} \dim(Y)+1}. \end{aligned}$$

□

The next ingredient is the localization formula due to Atiyah-Bott, and Berline-Vergne [1, 2].

Theorem 2.4. *Let the circle act on a compact manifold M . Fix a class $\alpha \in H_{S^1}^*(M; \mathbb{Q})$. Then as elements of $\mathbb{Q}(t)$,*

$$\int_M \alpha = \sum_{F \subset M^{S^1}} \int_F \frac{\alpha|_F}{e^{S^1}(N_F)},$$

where the sum is over all fixed point set components.

We often take a class α whose degree is less than the dimension of the manifold M , so the left hand side is 0, and for the integration of $\frac{\alpha|_F}{e^{S^1}(N_F)}$ on F , the only term which matters is the one containing the volume form of F .

In later sections, when we use Theorem 2.4, we will encounter a pure algebraic fact:

Lemma 2.5. *Let $n \in \mathbb{N}$, and A and B be respectively the coefficients of w^{n-1} and w^n in $(1 + w + w^2 + \cdots + w^n)^{n+1}$. Then*

$$A = \begin{cases} \binom{n+1}{1} + \binom{n+1}{2} \binom{n-1}{1} + \cdots + \binom{n+1}{\frac{n}{2}-1} \binom{n-1}{\frac{n}{2}-2} + \binom{n+1}{\frac{n}{2}} \binom{n-1}{\frac{n}{2}-1}, & \text{if } n \text{ is even,} \\ \binom{n+1}{1} + \binom{n+1}{2} \binom{n-1}{1} + \cdots + \binom{n+1}{\frac{n-1}{2}} \binom{n-1}{\frac{n-3}{2}} + \binom{n+1}{\frac{n+1}{2}} \binom{n-1}{\frac{n-1}{2}}, & \text{if } n \text{ is odd,} \end{cases}$$

and

$$B = \begin{cases} 2 \binom{n+1}{1} + 2 \binom{n+1}{2} \binom{n-1}{1} + \cdots + 2 \binom{n+1}{\frac{n}{2}-1} \binom{n-1}{\frac{n}{2}-2} + 2 \binom{n+1}{\frac{n}{2}} \binom{n-1}{\frac{n}{2}-1}, & \text{if } n \text{ is even,} \\ 2 \binom{n+1}{1} + 2 \binom{n+1}{2} \binom{n-1}{1} + \cdots + 2 \binom{n+1}{\frac{n-1}{2}} \binom{n-1}{\frac{n-3}{2}} + \binom{n+1}{\frac{n+1}{2}} \binom{n-1}{\frac{n-1}{2}}, & \text{if } n \text{ is odd.} \end{cases}$$

In particular, $B = 2A$.

Proof. By writing

$$(1 + w + w^2 + \cdots + w^n)^{n+1} = 1 + \binom{n+1}{1} (\sum_{i=1}^n w^i) + \binom{n+1}{2} (\sum_{i=1}^n w^i)^2 + \cdots + \binom{n+1}{n} (\sum_{i=1}^n w^i)^n + \text{higher order terms,}$$

and by only looking at the terms containing w^{n-1} and w^n , we get

$$A = \binom{n+1}{1} + \binom{n+1}{2} \binom{n-1}{1} + \binom{n+1}{3} \binom{n-1}{2} + \cdots + \binom{n+1}{n-1} \binom{n-1}{n-2},$$

and

$$B = \binom{n+1}{1} + \binom{n+1}{2} \binom{n-1}{1} + \binom{n+1}{3} \binom{n-1}{2} + \cdots + \binom{n+1}{n} \binom{n-1}{n-1}.$$

Then by grouping equal terms together, we get the claim. \square

We will only need that $B = 2A$. We write out the values of A and B for verification.

The last ingredient is the total Chern class, the total equivariant Chern class, and the equivariant Euler class of an S^1 -(almost) complex vector bundle over a compact manifold. We state the following lemma here to help us to understand the relations between these classes.

Lemma 2.6. [11, Lemma 2.4] *Let the circle act on a complex vector bundle E of complex rank d over a compact manifold X so that $E^{S^1} = X$. Assume that there exists a non-zero $\lambda \in \mathbb{Z}$ so that the circle acts on E with weight λ . Then there exists $c_i \in H^{2i}(X; \mathbb{Z})$ for all $0 \leq i \leq d$ such that*

$$\begin{aligned} c(E) &= 1 + c_1 + \cdots + c_{d-1} + c_d, \\ c^{S^1}(E) &= (1 + \lambda t)^d + c_1(1 + \lambda t)^{d-1} + \cdots + c_{d-1}(1 + \lambda t) + c_d, \text{ and} \\ e^{S^1}(E) &= (\lambda t)^d + c_1(\lambda t)^{d-1} + \cdots + c_{d-1}(\lambda t) + c_d. \end{aligned}$$

Here, $c(E)$, $c^{S^1}(E)$, and $e^{S^1}(E)$ are the total Chern class of E , the total equivariant Chern class of E , and the equivariant Euler class of E , respectively.

When we work with $e^{S^1}(N_X)$ in later sections, we will need the following algebraic fact:

Lemma 2.7. *Let $n > 1$ and $n \in \mathbb{N}$, $a_0, a_1, \dots, a_n \in \mathbb{Z}$, $m \in \mathbb{N}$, t and u be variables. If*

$$\begin{aligned} \left(t + \frac{a_0}{m}u\right)(t^n + a_1t^{n-1}u + \dots + a_nu^n) &= \left(t + \frac{u}{m}\right)^{n+1} \pmod{u^{n+1}}, \text{ or} \\ t\left(t + \frac{a_0}{m}u\right)(t^{n-1} + a_1t^{n-2}u + \dots + a_{n-1}u^{n-1}) &= \left(t + \frac{u}{m}\right)^{n+1} \pmod{u^{n+1}} \end{aligned}$$

holds, then $m = 1$.

Proof. Comparing the coefficients of tu^n on both sides of the equality, we get

$$m^{n-1} | (n+1).$$

So $m = 1$ if $n \geq 4$. For $n = 2$ and $n = 3$, if $m \neq 1$, then $m = 3$ and $m = 2$ respectively, for these two possibilities, comparing the coefficients containing u and u^2 on both sides of the equality, we see that they are not possible. Hence $m = 1$ for all $n \geq 2$. \square

3. PROOF OF THEOREM 1.5

In this section, we prove Theorem 1.5.

First, we prove two elementary lemmas.

Lemma 3.1. *Let (M, ω) be a compact symplectic manifold admitting a Hamiltonian S^1 action such that M^{S^1} consists of two connected components X and Y . If $\text{codim}(Y) > 2$, then $b_2(M) = b_2(X)$, and $[\omega]$ is primitive integral if and only if $[\omega|_X]$ is primitive integral. Similarly, if $\text{codim}(X) > 2$, then $b_2(M) = b_2(Y)$, and $[\omega]$ is primitive integral if and only if $[\omega|_Y]$ is primitive integral.*

Proof. Let ϕ be the moment map and assume $\phi(X) < \phi(Y)$. Using ϕ as a Morse-Bott function, $\text{codim}(Y)$ is the Morse index of Y . If $\text{codim}(Y) > 2$, then the restriction map $H^2(M; \mathbb{Z}) \rightarrow H^2(X; \mathbb{Z})$ is an isomorphism, so $b_2(M) = b_2(X)$, and $[\omega]$ is primitive integral if and only if $u = [\omega|_X]$ is primitive integral. Similarly, using $-\phi$, we get the other claims. \square

Lemma 3.2. *Let (M, ω) be a compact symplectic manifold admitting a Hamiltonian S^1 action such that M^{S^1} consists of two connected components X and Y with $\dim(X) + \dim(Y) = \dim(M)$. If $\dim(X) \geq \frac{1}{2} \dim(M)$ and $b_{2i}(X) = 1$ for all $0 \leq 2i \leq \dim(X)$, then $\dim(X) = \frac{1}{2} \dim(M)$.*

Proof. Let ϕ be the moment map and assume $\phi(X) < \phi(Y)$. Since ϕ is a perfect Morse-Bott function, we have

$$(3.3) \quad \dim H^i(M) = \dim H^i(X) + \dim H^{i-\text{codim}(Y)}(Y), \quad \forall i.$$

In our case, $\text{codim}(Y) = \dim(X) = 2k$ for some k . So $b_{2i}(X) = 1, \forall 0 \leq 2i \leq 2k$ implies that

$$(3.4) \quad b_{2i}(M) = 1, \quad \forall 0 \leq 2i < 2k, \text{ and } b_{2k}(M) = 2.$$

If $\dim(X) > \frac{1}{2} \dim(M)$, then $\dim(M) - 2k < 2k$. Since M is compact and oriented, by Poincaré duality, $b_{\dim(M)-2k}(M) = b_{2k}(M) = 2$, which contradicts (3.4). \square

The next result, Proposition 3.5, is a main step toward proving Theorem 1.5. It is also an important step for determining $e^{S^1}(N_X)$.

Proposition 3.5. *Let (M, ω) be a compact symplectic manifold of dimension bigger than 4 admitting a Hamiltonian S^1 action with moment map ϕ such that M^{S^1} consists of two connected components X and Y with $\dim(X) + \dim(Y) = \dim(M)$ and $\phi(X) < \phi(Y)$. Assume $[\omega]$ is a primitive integral class, $2n = \dim(X) \geq \frac{1}{2} \dim(M)$, and $H^{\text{even}}(X; \mathbb{Z}) = \mathbb{Z}[u]/u^{n+1}$, where $u = [\omega|_X]$. If the action is semifree, then*

$$\phi(Y) - \phi(X) = 1,$$

and there exists $a_0 \in \mathbb{Z}$ such that

$$(t + a_0 u) e^{S^1}(N_X) = (t + u)^{n+1}.$$

Proof. By the assumptions and Lemma 3.2, $2n = \dim(X) = \frac{1}{2} \dim(M) = \dim(Y) > 2$. Since $\text{codim}(Y) = \dim(X) > 2$, by Lemma 3.1, $u = [\omega|_X]$ is primitive integral.

Since $[\omega]$ is an integral class, $\phi(Y) - \phi(X) = m \in \mathbb{N}$.

By Lemma 2.2, there exists $\lambda \in H_{S^1}^*(X; \mathbb{Z})$ such that

$$(3.6) \quad \lambda e^{S^1}(N_X) = (mt + u)^{n+1}.$$

Since the action is semifree, and $\text{rank}_{\mathbb{C}}(N_X) = \frac{1}{2} \dim(Y) = n$, by Lemma 2.6, and the assumption $H^{\text{even}}(X; \mathbb{Z}) = \mathbb{Z}[u]/u^{n+1}$, we have

$$e^{S^1}(N_X) = t^n + a_1 t^{n-1} u + \cdots + a_n u^n, \text{ where } a_i \in \mathbb{Z}, \forall i.$$

By degree reasons and by comparing the coefficients of t^{n+1} on both sides of (3.6), we may let

$$\lambda = m^{n+1} t + au, \text{ with } a \in \mathbb{Z}.$$

Comparing the coefficients of $t^n u$ on both sides of (3.6), we get

$$a = a_0 m^n \text{ for some } a_0 \in \mathbb{Z}.$$

So we may write (3.6) as

$$(3.7) \quad \left(t + \frac{a_0}{m} u\right) (t^n + a_1 t^{n-1} u + \cdots + a_n u^n) = \left(t + \frac{u}{m}\right)^{n+1} \mod u^{n+1}.$$

By Lemma 2.7, $m = 1$. Both claims follow. \square

To prove Theorem 1.5, let us also recall the following results.

Lemma 3.8. [9, Lemma 2.3] *Let the circle act on a connected compact symplectic manifold (M, ω) with moment map $\phi: M \rightarrow \mathbb{R}$. Assume $b_2(M) = 1$. Then*

$$c_1(M) = \frac{\Gamma_F - \Gamma_{F'}}{\phi(F') - \phi(F)} [\omega],$$

where F and F' are any two fixed components such that $\phi(F') \neq \phi(F)$.

Proposition 3.9. [11, Proposition 7.5] *Let (M, ω) be a compact symplectic manifold admitting an effective Hamiltonian S^1 action with moment map ϕ such that M^{S^1} consists of two connected components X and Y with $\phi(X) < \phi(Y)$. Then the set of distinct weights of the S^1 action on the normal bundles N_X of X and N_Y of Y are respectively $\{1, 2, \dots, N\}$ and $\{-1, -2, \dots, -N\}$.*

Now we are ready to prove Theorem 1.5.

Proof of Theorem 1.5. With no loss of generality, assume $\phi(X) < \phi(Y)$.

By Lemma 3.2, $\dim(X) = \frac{1}{2} \dim(M)$. The fact $\text{codim}(Y) > 2$ and Lemma 3.1 imply that $b_2(M) = b_2(X) = 1$.

By Proposition 3.5, (1) \implies (2). Conversely, if there exists any nontrivial finite stabilizer, then by Lemma 2.1, $\phi(Y) - \phi(X) > 1$. Hence (2) \implies (1).

If the action is semifree, then $\Gamma_X = \text{rank}_{\mathbb{C}}(N_X) = \frac{1}{2} \dim(Y)$, and similarly $\Gamma_Y = -\frac{1}{2} \dim(X)$. Moreover, by the last step, $\phi(Y) - \phi(X) = 1$. Then (3) follows from Lemma 3.8.

To prove (3) \implies (1), we only need $b_2(X) = b_2(M) = 1$. Assume the action is not semifree. By Propositions 4.1 and 3.9, $\dim(X) = \dim(Y)$, and the set of distinct weights on N_X is $\{1, 2, \dots, N\}$. Let m_i be the multiplicity of the weight i on N_X . Then $\sum_i m_i = \text{rank}_{\mathbb{C}}(N_X) = \frac{1}{2} \dim(Y)$, and

$$\Gamma_X = Nm_N + (N-1)m_{N-1} + \dots + 1 \cdot m_1 < N \frac{1}{2} \dim(Y), \text{ and } \Gamma_Y = -\Gamma_X.$$

By Lemma 2.1, $\phi(Y) - \phi(X) \geq N \cdot (N-1)$. By Lemma 3.8,

$$c_1(M) = \frac{2\Gamma_X}{\phi(Y) - \phi(X)}[\omega] < \dim(Y)[\omega] = \frac{1}{2} \dim(M)[\omega].$$

□

4. DETERMINING THE ACTION

In this section, under a cohomology condition on both fixed point set components, we prove that non-semifree actions do not exist.

First, let us look at what stabilizer groups can occur.

Proposition 4.1. [11, Lemma 7.1 and Proposition 7.9] *Let (M, ω) be a compact symplectic manifold admitting an effective Hamiltonian S^1 action with M^{S^1} consisting of two connected components X and Y . Then*

- (1) *if the action is not semifree, then $\dim(X) = \dim(Y)$, and*
- (2) *if additionally, $b_{2i}(X) = 1$ for all $0 \leq 2i \leq \dim(X)$, then the only finite stabilizer groups are 1 and \mathbb{Z}_2 , and $\dim(M^{\mathbb{Z}_2}) - \dim(X) = 2$ or $\dim(M) - \dim(M^{\mathbb{Z}_2}) = 2$ or both.*

The next fact was implied by [11, Lemma 7.6], but was stated differently. We state it as follows, and will use it twice.

Lemma 4.2. *Let (M, ω) be a compact symplectic manifold admitting a Hamiltonian S^1 action such that M^{S^1} consists of two connected components X and Y such that $\dim(M) - \dim(X) = \dim(M) - \dim(Y) = 2$. Assume $b_2(X) = b_2(Y) = 1$, $[\omega|_X]$ and $[\omega|_Y]$ are both primitive integral. Then $c_1(N_X) = c_1(N_Y) = 0$.*

Next, in Lemmas 4.3 and 4.9, assuming a cohomology condition on one fixed set component, we determine the equivariant Euler class of its normal bundle.

Lemma 4.3. *Let (M, ω) be a compact symplectic manifold admitting an effective non-semifree Hamiltonian S^1 action with moment map ϕ such that M^{S^1} consists of two connected components X and Y with $\dim(X) + \dim(Y) = \dim(M)$ and $\phi(X) < \phi(Y)$. Assume $[\omega]$ is a primitive integral class, and $H^{\text{even}}(X; \mathbb{Z}) = \mathbb{Z}[u]/u^{n+1}$, where $u = [\omega|_X]$ and $2n = \dim(X)$. Then $\dim(X) = \dim(Y) = 2n \geq 6$, the only finite stabilizer groups are 1 and \mathbb{Z}_2 , and*

$$(4.4) \quad \dim(M^{\mathbb{Z}_2}) - \dim(X) > 2, \quad \dim(M) - \dim(M^{\mathbb{Z}_2}) = 2,$$

$$(4.5) \quad \phi(Y) - \phi(X) = 2, \quad \text{and} \quad e^{S^1}(N_{M^{\mathbb{Z}_2}})|_X = t + u.$$

Proof. Since the action is not semifree, by Proposition 4.1, $\dim(X) = \dim(Y) = \text{codim}(Y) = 2n$ for some $n \geq 2$. By Lemma 3.1, $b_2(X) = b_2(M) = b_2(Y) = 1$, $u = [\omega|_X]$ and $v = [\omega|_Y]$ are primitive integral. After (4.4) is shown, we get $2n \geq 6$.

Since $b_{2i}(X) = 1$ for all $0 \leq 2i \leq 2n$, by Proposition 4.1, the only finite stabilizer groups are 1 and \mathbb{Z}_2 . Since $[\omega]$ is integral, $m = \phi(Y) - \phi(X) \in \mathbb{N}$. Since there is \mathbb{Z}_2 stabilizer, by Lemma 2.1,

$$(4.6) \quad 2 \mid m.$$

Assume instead $\dim(M^{\mathbb{Z}_2}) - \dim(X) = 2$. Then by Lemmas 4.2 and 2.6,

$$e^{S^1}(N_X^{M^{\mathbb{Z}_2}}) = 2t,$$

where $N_X^{M^{\mathbb{Z}_2}}$ is the normal bundle of X in $M^{\mathbb{Z}_2}$. The action on $N_{M^{\mathbb{Z}_2}}|_X$ is semifree, and $\text{rank}_{\mathbb{C}}(N_{M^{\mathbb{Z}_2}}|_X) = n - 1$. Since $H^{\text{even}}(X; \mathbb{Z}) = \mathbb{Z}[u]/u^{n+1}$, by Lemma 2.6, we may write

$$e^{S^1}(N_{M^{\mathbb{Z}_2}})|_X = t^{n-1} + a_1 t^{n-2} u + \cdots + a_{n-1} u^{n-1}, \quad \text{with } a_i \in \mathbb{Z}, \forall i.$$

We have

$$(4.7) \quad e^{S^1}(N_X) = e^{S^1}(N_X^{M^{\mathbb{Z}_2}}) e^{S^1}(N_{M^{\mathbb{Z}_2}})|_X.$$

By Lemma 2.2, there exist $a, b \in \mathbb{Z}$ such that

$$(at + bu) \cdot 2t \cdot (t^{n-1} + a_1 t^{n-2} u + \cdots + a_{n-1} u^{n-1}) = (mt + u)^{n+1}.$$

Comparing the coefficients of t^{n+1} and $t^n u$ on both sides, we get that there exists $a_0 \in \mathbb{Z}$ such that

$$t \left(t + \frac{a_0}{m} u \right) (t^{n-1} + a_1 t^{n-2} u + \cdots + a_{n-1} u^{n-1}) = \left(t + \frac{u}{m} \right)^{n+1} \mod u^{n+1}.$$

By Lemma 2.7, $m = 1$, which contradicts (4.6). Together with Proposition 4.1, (4.4) follows.

Since $\dim(M^{\mathbb{Z}_2}) - \dim(X) > 2$ and $b_2(X) = 1$, by [11, Lemma 7.7],

$$(4.8) \quad c_1(N_{M^{\mathbb{Z}_2}})|_X = 2 \frac{\Gamma_1}{m} u = \frac{2}{m} u,$$

where Γ_1 is the sum of the weights 1's on the normal bundle to X , which is 1 here. Since $\frac{2}{m}$ needs to be an integer, we have $m \mid 2$. Together with (4.6), (4.8), and that $N_{M^{\mathbb{Z}_2}}|_X$ is a complex line bundle and the weight of the action on it is 1, (4.5) follows. \square

Lemma 4.9. *Assume the assumptions of Lemma 4.3 hold. Then $\dim(X) = \dim(Y) = 2n$ with $n \geq 3$ being odd, and*

$$4t e^{S^1}(N_X) = (2t + u)^{n+1}.$$

Proof. By Lemma 4.3, $\dim(X) = \dim(Y) = 2n \geq 6$, and

$$\phi(Y) - \phi(X) = 2, \text{ and } e^{S^1}(N_{M^{\mathbb{Z}_2}})|_X = t + u.$$

By (4.4) and Lemma 2.6, we may write

$$e^{S^1}(N_X^{M^{\mathbb{Z}_2}}) = (2t)^{n-1} + a_1(2t)^{n-2}u + \cdots + a_{n-1}u^{n-1}, \text{ with } a_i \in \mathbb{Z}, \forall i.$$

By (4.7) and Lemma 2.2, there exist $c, d \in \mathbb{Z}$ such that

$$(ct + du)(t + u)((2t)^{n-1} + a_1(2t)^{n-2}u + \cdots + a_{n-1}u^{n-1}) = (2t + u)^{n+1}.$$

Comparing the coefficients of t^{n+1} and of $t^n u$ on both sides, we get $c = 4$, and $d = 2a_0$ for some $a_0 \in \mathbb{Z}$. Hence

$$(4.10) \quad (2t + a_0 u)(2t + 2u)((2t)^{n-1} + a_1(2t)^{n-2}u + \cdots + a_{n-1}u^{n-1}) \\ = (2t + u)^{n+1} \mod u^{n+1}.$$

So there exists $\lambda \in \mathbb{C}$ such that $(2t + u)^{n+1} + (\lambda u)^{n+1}$, as a polynomial, is equal to the left hand side of (4.10). We have $1 + \lambda^{n+1} = 2a_0 a_{n-1}$ (by comparing the coefficients of u^{n+1} on both sides), and

$$(2t + u)^{n+1} + (\lambda u)^{n+1} = \prod_{k=0}^n \left(2t + u + e^{\frac{2\pi k \sqrt{-1}}{n+1}} \lambda u \right).$$

Since there is a linear factor $2t + 2u$ on the left hand side of (4.10), there exists a k such that $e^{\frac{2\pi k \sqrt{-1}}{n+1}} \lambda u = u$. So $|\lambda| = 1$. Since there is another linear factor $2t + a_0 u$ on the left hand side of (4.10), there is another k' such that $e^{\frac{2\pi k' \sqrt{-1}}{n+1}} \lambda u$ is real and it must be $-u$. Hence n must be odd, and the linear factor $2t + a_0 u = 2t$. \square

Now, we reach our final result of this section:

Proposition 4.11. *There exists no compact symplectic manifold (M, ω) admitting an effective non-semifree Hamiltonian S^1 action such that M^{S^1} consists of two connected components X and Y satisfying $\dim(X) + \dim(Y) = \dim(M)$, $H^{even}(X; \mathbb{Z}) = \mathbb{Z}[u]/u^{\frac{1}{2}\dim(X)+1}$ and $H^{even}(Y; \mathbb{Z}) = \mathbb{Z}[v]/v^{\frac{1}{2}\dim(Y)+1}$, where $u = [\omega|_X]$, $v = [\omega|_Y]$ and $[\omega]$ is integral.*

Proof. Assume such a symplectic manifold (M, ω) exists, the moment map is ϕ and $\phi(X) < \phi(Y)$. By Lemma 4.3, $\dim(X) = \dim(Y) = 2n$ for some $n \geq 3$. We may assume $[\omega]$ is primitive. By Lemma 4.9,

$$e^{S^1}(N_X) = \frac{(2t + u)^{n+1}}{4t}.$$

Similarly, by symmetry,

$$e^{S^1}(N_Y) = \frac{(-2t + v)^{n+1}}{-4t}.$$

Using Theorem 2.4 to integrate 1 on M , we get

$$0 = \int_X \frac{1}{e^{S^1}(N_X)} + \int_Y \frac{1}{e^{S^1}(N_Y)},$$

from which, we get

$$(4.12) \quad 0 = \int_X \frac{1}{(2t + u)^{n+1}} + (-1)^n \int_Y \frac{1}{(2t - v)^{n+1}}.$$

Let $w = -u$, then $w^n = (-1)^n u^n$, so $\int_X w^n = (-1)^n$. From (4.12), we get

$$(4.13) \quad 0 = \int_X \frac{1}{\left(1 - \frac{w}{2t}\right)^{n+1}} + (-1)^n \int_Y \frac{1}{\left(1 - \frac{v}{2t}\right)^{n+1}}.$$

We have

$$\frac{1}{\left(1 - \frac{w}{2t}\right)^{n+1}} = \left(1 + \frac{w}{2t} + \cdots + \left(\frac{w}{2t}\right)^n\right)^{n+1}.$$

Let A be the coefficient of $\left(\frac{w}{2t}\right)^n$ in this expression. Then $A > 0$, and A is also the coefficient of $\left(\frac{v}{2t}\right)^n$ in $\frac{1}{\left(1 - \frac{v}{2t}\right)^{n+1}}$. Then (4.13) gives

$$0 = (-1)^n A + (-1)^n A, \text{ a contradiction.}$$

□

5. THE EQUIVARIANT EULER CLASS OF THE NORMAL BUNDLE OF THE FIXED POINT SET

In this section, assuming a cohomology condition on both fixed set components, we determine the equivariant Euler class of the normal bundle of the fixed point set.

Proposition 5.1. *Let (M, ω) be a compact symplectic manifold admitting an effective Hamiltonian S^1 action with moment map ϕ such that M^{S^1} consists of two connected components X and Y with $\dim(X) + \dim(Y) = \dim(M)$ and $\phi(X) < \phi(Y)$. Assume $[\omega]$ is primitive integral,*

$$H^{even}(X; \mathbb{Z}) = \mathbb{Z}[u]/u^{\frac{1}{2}\dim(X)+1} \text{ and } H^{even}(Y; \mathbb{Z}) = \mathbb{Z}[v]/v^{\frac{1}{2}\dim(Y)+1},$$

where $u = [\omega|_X]$ and $v = [\omega|_Y]$. Then the action must be semifree, $\dim(X) = \dim(Y) = 2n$ with $n \geq 1$,

$$(5.2) \quad e^{S^1}(N_X) = \frac{(t+u)^{n+1}}{t+2u}, \text{ and } e^{S^1}(N_Y) = \frac{(-t+v)^{n+1}}{-t+2v}.$$

Proof. By Proposition 4.11, the action must be semifree.

Since $\dim(X) + \dim(Y) = \dim(M)$, we have $\dim(X) \geq \frac{1}{2}\dim(M)$ or $\dim(Y) \geq \frac{1}{2}\dim(M)$. By Lemma 3.2, we get $\dim(X) = \dim(Y) = 2n = \frac{1}{2}\dim(M)$ for some $n \geq 1$.

First, assume $n = 1$, i.e., $\dim(X) = \dim(Y) = 2$ and $\dim(M) = 4$. In this case $\text{rank}_{\mathbb{C}}(N_X) = \text{rank}_{\mathbb{C}}(N_Y) = 1$. The assumption $H^2(X; \mathbb{Z}) = \mathbb{Z}[u]/u^2$ and $H^2(Y; \mathbb{Z}) = \mathbb{Z}[v]/v^2$ means that u and v are both primitive integral. By Lemma 4.2,

$$e^{S^1}(N_X) = t + 0u, \text{ and } e^{S^1}(N_Y) = -t + 0v.$$

Hence (5.2) holds for dimension 4.

Next, assume $n > 1$. By Proposition 3.5,

$$e^{S^1}(N_X) = \frac{(t+u)^{n+1}}{t+au}, \text{ with } a \in \mathbb{Z}.$$

Similarly, by symmetry,

$$e^{S^1}(N_Y) = \frac{(-t+v)^{n+1}}{-t+bv}, \text{ with } b \in \mathbb{Z}.$$

Using Theorem 2.4 to Integrate 1 on M , we get

$$\int_X \frac{1}{e^{S^1}(N_X)} + \int_Y \frac{1}{e^{S^1}(N_Y)} = 0,$$

from which we get

$$\int_X \frac{t+au}{(t+u)^{n+1}} + (-1)^n \int_Y \frac{t-bv}{(t-v)^{n+1}} = 0.$$

Let $w = -u$, then $w^n = (-1)^n u^n$, and $\int_X w^n = (-1)^n$. The above integral becomes

$$\int_X \frac{t-aw}{(1-\frac{w}{t})^{n+1}} + (-1)^n \int_Y \frac{t-bv}{(1-\frac{v}{t})^{n+1}} = 0.$$

Writing $(1-\frac{w}{t})^{n+1} = (1+\frac{w}{t}+\dots+(\frac{w}{t})^n)^{n+1}$, and let A and B be respectively the coefficients of $(\frac{w}{t})^{n-1}$ and $(\frac{w}{t})^n$ in this expression. Then the above integral gives

$$(-1)^n(B-aA) + (-1)^n(B-bA) = 0.$$

By Lemma 2.5, $B = 2A \neq 0$. So

$$a + b = 4.$$

Next, consider integrating $c_1^{S^1}(M)$ on M . By Theorem 1.5, $c_1(M)|_X = 2nu$ and $c_1(M)|_Y = 2nv$. Moreover, $\Gamma_X = n$ and $\Gamma_Y = -n$. So we have

$$c_1^{S^1}(M)|_X = nt + 2nu, \text{ and } c_1^{S^1}(M)|_Y = -nt + 2nv.$$

Since $\dim(M) > 2$, we have

$$\int_X \frac{c_1^{S^1}(M)|_X}{e^{S^1}(N_X)} + \int_Y \frac{c_1^{S^1}(M)|_Y}{e^{S^1}(N_Y)} = 0.$$

Using the same trick as the above (we do not need Lemma 2.5 this time, only the sign and the same expression in w and v matter), it is not hard to see that

$$a = b.$$

Hence $a = b = 2$, and (5.2) follows. \square

6. THE INTEGRAL COHOMOLOGY RING OF THE FIXED POINT SET

In this section, for $\dim(M) > 4$, with the cohomology condition in even degrees on both fixed set components, we find the ring $H^*(X; \mathbb{Z})$ and $H^*(Y; \mathbb{Z})$.

Lemma 6.1. *Let (M, ω) be a compact symplectic manifold of dimension bigger than 4 admitting a Hamiltonian S^1 action such that M^{S^1} consists of two connected components X and Y with $\dim(X) + \dim(Y) = \dim(M)$. Assume $[\omega]$ is primitive integral,*

$$H^{even}(X; \mathbb{Z}) = \mathbb{Z}[u]/u^{\frac{1}{2}\dim(X)+1} \text{ and } H^{even}(Y; \mathbb{Z}) = \mathbb{Z}[v]/v^{\frac{1}{2}\dim(Y)+1},$$

where $u = [\omega|_X]$ and $v = [\omega|_Y]$. Then $\dim(X) = \dim(Y) = 2n > 2$,

$$H^*(X; \mathbb{Z}) = \mathbb{Z}[u]/u^{n+1} \text{ and } H^*(Y; \mathbb{Z}) = \mathbb{Z}[v]/v^{n+1}.$$

Proof. Let ϕ be the moment map and assume $\phi(X) < \phi(Y)$. By quotienting out a finite subgroup action if necessary, we may assume the action is effective. By Proposition 5.1, the action is semifree, $\dim(X) = \dim(Y) = 2n > 2$ (we assumed $\dim(M) > 4$), and

$$\begin{aligned} e^{S^1}(N_Y) &= (-t)^n + (n-1)v(-t)^{n-1} + \text{lower order terms} \\ &= (-1)^n(t^n + (1-n)vt^{n-1} + \text{lower order terms}). \end{aligned}$$

By [11, Lemma 8.5], if there exists a class \tilde{u} such that $\tilde{u}|_x = 0$, $\forall x \in X$ and $\tilde{u}|_y \neq 0$, $\forall y \in Y$, and there exists a class $\tilde{\mu}$ such that $\tilde{\mu}|_Y = (1-n)v$ (the degree 2 term, the coefficient of t^{n-1} in $e^{S^1}(N_Y)$ above) and $\tilde{\mu}|_x \neq -nt$, $\forall x \in X$, then

$$H^{2k+1}(X; \mathbb{R}) = 0 \text{ and } H^{2k+1}(X; \mathbb{Z}_p) = 0$$

for all k and all prime numbers p . By Lemma 2.1, the required \tilde{u} exists. Note that by Theorem 1.5, $\phi(Y) - \phi(X) = 1$. We may take

$$\tilde{\mu} = (1 - n)\tilde{u} + (1 - n)(\phi(Y) - \phi(X))t = (1 - n)\tilde{u} + (1 - n)t.$$

Hence $H^*(X; \mathbb{Z})$ has no torsion ([11, Lemma 4.7]), and no odd degree terms, and $H^*(X; \mathbb{Z}) = \mathbb{Z}[u]/u^{n+1}$ holds. Similarly, the claim holds for Y . \square

7. THE TOTAL CHERN CLASSES OF THE FIXED POINT SETS AND OF THEIR NORMAL BUNDLES

In this section, we compute $c(N_X)$, $c(N_Y)$, $c(X)$ and $c(Y)$.

First, Proposition 5.1 and Lemma 2.6 give us $c(N_X)$ and $c(N_Y)$.

Lemma 7.1. *Let (M, ω) be a compact symplectic manifold admitting a Hamiltonian S^1 action such that M^{S^1} consists of two connected components X and Y with $\dim(X) + \dim(Y) = \dim(M)$. Assume $[\omega]$ is primitive integral,*

$$H^{even}(X; \mathbb{Z}) = \mathbb{Z}[u]/u^{\frac{1}{2}\dim(X)+1} \text{ and } H^{even}(Y; \mathbb{Z}) = \mathbb{Z}[v]/v^{\frac{1}{2}\dim(Y)+1},$$

where $u = [\omega|_X]$ and $v = [\omega|_Y]$. Then $\dim(X) = \dim(Y) = 2n$ with $n \geq 1$,

$$c(N_X) = \frac{(1+u)^{n+1}}{1+2u}, \text{ and } c(N_Y) = \frac{(1+v)^{n+1}}{1+2v}.$$

To get $c(X)$ and $c(Y)$, we will use Proposition 5.1, Lemma 6.1 and the following result.

Proposition 7.2. [11, Prop. 5.1 and Cor. 5.2] *Let (M, ω) be a compact symplectic manifold admitting a Hamiltonian S^1 action with moment map ϕ such that M^{S^1} consists of two connected components X and Y . Assume the action is semifree or $H^*(M^{S^1}; \mathbb{Z})$ has no torsion. Then there is an isomorphism*

$$f: H_{S^1}^*(X; \mathbb{Z})/e^{S^1}(N_X) \rightarrow H_{S^1}^*(Y; \mathbb{Z})/e^{S^1}(N_Y) \text{ such that}$$

$$f(\tilde{\alpha}|_X) = \tilde{\alpha}|_Y, \quad \forall \tilde{\alpha} \in H_{S^1}^*(M; \mathbb{Z}).$$

In particular, if $[\omega]$ is integral, then

$$f([\omega|_X]) = [\omega|_Y] + t(\phi(X) - \phi(Y)).$$

Proposition 7.3. *Let (M, ω) be a compact symplectic manifold of dimension bigger than 4 admitting a Hamiltonian S^1 action such that M^{S^1} consists of two connected components X and Y with $\dim(X) + \dim(Y) = \dim(M)$. Assume $[\omega]$ is primitive integral,*

$$H^{even}(X; \mathbb{Z}) = \mathbb{Z}[u]/u^{\frac{1}{2}\dim(X)+1} \text{ and } H^{even}(Y; \mathbb{Z}) = \mathbb{Z}[v]/v^{\frac{1}{2}\dim(Y)+1},$$

where $u = [\omega|_X]$ and $v = [\omega|_Y]$. Then $\dim(X) = \dim(Y) = 2n > 2$,

$$(7.4) \quad c(X) = (1+u)^{n+1}, \text{ and } c(Y) = (1+v)^{n+1}.$$

Proof. Let ϕ be the moment map, and assume $\phi(X) < \phi(Y)$.

We may assume the action is effective. By Proposition 5.1, the action is semifree, $\dim(X) = \dim(Y) = 2n > 2$, and together with Lemma 2.6, we have

$$(7.5) \quad c^{S^1}(N_X) = \frac{(1+t+u)^{n+1}}{1+t+2u}, \text{ and } c^{S^1}(N_Y) = \frac{(1-t+v)^{n+1}}{1-t+2v}.$$

By Theorem 1.5,

$$(7.6) \quad \phi(Y) - \phi(X) = 1.$$

Since $H^{\text{even}}(X; \mathbb{Z}) = \mathbb{Z}[u]/u^{n+1}$, we may assume

$$(7.7) \quad c(X) = 1 + a_1u + \cdots + a_iu^i + \cdots + a_nu^n, \text{ with } a_i \in \mathbb{Z} \text{ for } 1 \leq i \leq n.$$

Moreover, by Lemma 6.1, $H^{\text{odd}}(X; \mathbb{R}) = 0$. So by the Euler characteristic formula, we first get

$$a_n = \int_X a_n u^n = \sum_{i=0}^n (-1)^i \dim H^i(X; \mathbb{R}) = n + 1.$$

Next, we compute the other a_i 's. We have that

$$(7.8) \quad c^{S^1}(M)|_X = c^{S^1}(N_X)c(X), \text{ and } c^{S^1}(M)|_Y = c^{S^1}(N_Y)c(Y),$$

and we may similarly write

$$(7.9) \quad c(Y) = 1 + \sum_{i=1}^n b_i v^i \text{ with } b_i \in \mathbb{Z}.$$

Consider the map f in Proposition 7.2 composed with the restriction map $H_{S^1}^*(Y; \mathbb{Z})/e^{S^1}(N_Y) \rightarrow H_{S^1}^*(y; \mathbb{Z})/(-t)^n$, where $y \in Y$ is a point,

$$g: H_{S^1}^*(X; \mathbb{Z})/e^{S^1}(N_X) \rightarrow H_{S^1}^*(Y; \mathbb{Z})/e^{S^1}(N_Y) \rightarrow H_{S^1}^*(y; \mathbb{Z})/(-t)^n.$$

First, by (7.6), we have

$$g(u) = (v + (\phi(X) - \phi(Y))t)|_y = -t.$$

Together with (7.5), (7.7), (7.8), and (7.9), we obtain

$$g(c^{S^1}(M)|_X) = \frac{1 + \sum_{i=1}^n a_i (-t)^i}{1-t} = (c^{S^1}(M)|_Y)|_y = (1-t)^n \pmod{t^n},$$

which gives

$$1 + \sum_{i=1}^n a_i (-t)^i = (1-t)^{n+1} \pmod{t^n}.$$

From this formula, we can get a_i for $1 \leq i \leq n-1$. Together with the value of a_n and (7.7), we get (7.4) for $c(X)$. The claim for $c(Y)$ follows similarly. \square

8. THE INTEGRAL COHOMOLOGY RING AND TOTAL CHERN CLASS OF M AND THE PROOF OF THEOREM 1.6

In this section, we first obtain $H_{S^1}^*(M; \mathbb{Z})$, $H^*(M; \mathbb{Z})$, $c^{S^1}(M)$ and $c(M)$, then we prove Theorem 1.6.

Proposition 8.1. *Let (M, ω) be a compact symplectic manifold of dimension bigger than 4 admitting an effective Hamiltonian S^1 action with moment map ϕ such that M^{S^1} consists of two connected components X and Y with $\dim(X) + \dim(Y) = \dim(M)$. Assume $[\omega]$ is primitive integral,*

$$H^{even}(X; \mathbb{Z}) = \mathbb{Z}[u]/u^{\frac{1}{2}\dim(X)+1} \quad \text{and} \quad H^{even}(Y; \mathbb{Z}) = \mathbb{Z}[v]/v^{\frac{1}{2}\dim(Y)+1},$$

where $u = [\omega|_X]$ and $v = [\omega|_Y]$. Then $\dim(X) = \dim(Y) = 2n > 2$,

$$c^{S^1}(M) = \frac{(1 + \tilde{u})^{n+1}(1 + t + \tilde{u})^{n+1}}{1 + t + 2\tilde{u}}, \quad \text{and} \quad c(M) = \frac{(1 + [\omega])^{2n+2}}{1 + 2[\omega]}.$$

Moreover, a basis of $H_{S^1}^*(M; \mathbb{Z})$ as an $H^*(\mathbb{CP}^\infty; \mathbb{Z})$ -module is:

$$\tilde{u}^i \in H_{S^1}^{2i}(M; \mathbb{Z}), \quad \text{and} \quad \frac{\tilde{u}^{n+1}(t + \tilde{u})^i}{t + 2\tilde{u}} \in H_{S^1}^{2n+2i}(M; \mathbb{Z}), \quad \text{where } 0 \leq i \leq n,$$

and

$$H^*(M; \mathbb{Z}) = \begin{cases} \mathbb{Z}[x, y]/(x^{n+1} - 2xy, y^2), & \text{if } n \text{ is odd,} \\ \mathbb{Z}[x, y]/(x^{n+1} - 2xy, y^2 - x^ny), & \text{if } n \text{ is even,} \end{cases}$$

where $x = [\omega]$ and $\deg(y) = 2n$. In the above, \tilde{u} is the class in Lemma 2.1.

Proof. By Proposition 5.1 and Theorem 1.5, the action is semifree, $\dim(X) = \dim(Y) = 2n > 2$, and

$$\phi(Y) - \phi(X) = 1.$$

First, we compute $c^{S^1}(M)$ and $c(M)$. We have

$$c^{S^1}(M)|_X = c(X)c^{S^1}(N_X) \quad \text{and} \quad c^{S^1}(M)|_Y = c(Y)c^{S^1}(N_Y).$$

By Propositions 7.3, 5.1, Lemmas 2.6 and 2.1, we have

$$c^{S^1}(M)|_X = (1 + u)^{n+1} \frac{(1 + t + u)^{n+1}}{1 + t + 2u} = (1 + \tilde{u})^{n+1} \frac{(1 + t + \tilde{u})^{n+1}}{1 + t + 2\tilde{u}}|_X,$$

and

$$c^{S^1}(M)|_Y = (1 + v)^{n+1} \frac{(1 - t + v)^{n+1}}{1 - t + 2v} = (1 + \tilde{u})^{n+1} \frac{(1 + t + \tilde{u})^{n+1}}{1 + t + 2\tilde{u}}|_Y.$$

Since the action is semifree, the map

$$(8.2) \quad H_{S^1}^*(M; \mathbb{Z}) \rightarrow H_{S^1}^*(M^{S^1}; \mathbb{Z})$$

induced by the inclusion is injective. Hence $c^{S^1}(M)$ is as claimed. Since by Lemma 6.1, $H^*(M^{S^1}; \mathbb{Z})$ is torsion free, we have

$$(8.3) \quad H^*(M; \mathbb{Z}) = H_{S^1}^*(M; \mathbb{Z})/(t).$$

So $c(M) = c^{S^1}(M)/(t)$ and it is as claimed. For the injectivity of (8.2) and the claim (8.3), we refer to [6] and [11, Sec. 2].

Next, we find a basis for $H_{S^1}^*(M; \mathbb{Z})$ as an $H^*(\mathbb{CP}^\infty; \mathbb{Z})$ -module. First, by Lemma 6.1,

$$H^*(X; \mathbb{Z}) = \mathbb{Z}[u]/u^{n+1} \quad \text{and} \quad H^*(Y; \mathbb{Z}) = \mathbb{Z}[v]/v^{n+1}.$$

By the method given in [6, 16], a way of getting a basis of $H_{S^1}^*(M; \mathbb{Z})$ as an $H^*(\mathbb{CP}^\infty; \mathbb{Z})$ -module is as follows. For the basis $\{1, u, \dots, u^n\}$ of $H^*(X; \mathbb{Z})$ and the basis $\{1, v, \dots, v^n\}$ of $H^*(Y; \mathbb{Z})$, there exists a basis of $H_{S^1}^*(M; \mathbb{Z})$ as an $H^*(\mathbb{CP}^\infty; \mathbb{Z})$ -module:

$$\alpha_i \in H_{S^1}^{2i}(M; \mathbb{Z}) \text{ such that } \alpha_i|_X = u^i, \text{ and}$$

$$\beta_i \in H_{S^1}^{2n+2i}(M; \mathbb{Z}) \text{ such that } \beta_i|_X = 0, \text{ and } \beta_i|_Y = v^i e^{S^1}(N_Y), \text{ where } 0 \leq i \leq n.$$

Since $\tilde{u}^i|_X = u^i$, we may take $\alpha_i = \tilde{u}^i$, $\forall 0 \leq i \leq n$. Next, we find the β_i 's.

By Proposition 5.1,

$$e^{S^1}(N_Y) = \frac{(-t+v)^{n+1}}{-t+2v}.$$

We have $(\tilde{u}+t)^i|_Y = v^i$. Notice that for $\forall 0 \leq i \leq n$,

$$(\tilde{u}^{n+1}(\tilde{u}+t)^i)|_X = ((2\tilde{u}+t)\beta_i)|_X = 0, \text{ and } (\tilde{u}^{n+1}(\tilde{u}+t)^i)|_Y = ((2\tilde{u}+t)\beta_i)|_Y.$$

By the injectivity of (8.2), we get

$$(8.4) \quad \tilde{u}^{n+1}(\tilde{u}+t)^i = (2\tilde{u}+t)\beta_i, \quad \forall 0 \leq i \leq n.$$

Hence we can express the β_i 's as claimed.

Finally, we will find the ring $H^*(M; \mathbb{Z})$. By (8.3), the image of the basis $\{\alpha_i, \beta_i \mid 0 \leq i \leq n\}$ under the restriction map

$$r: H_{S^1}^*(M; \mathbb{Z}) \rightarrow H^*(M; \mathbb{Z})$$

is a basis of $H^*(M; \mathbb{Z})$. We have

$$r(\alpha_i) = [\omega]^i = x^i, \quad \forall 0 \leq i \leq n, \text{ where } x = [\omega].$$

Let

$$r(\beta_0) = y.$$

Then a basis of $H^*(M; \mathbb{Z})$ is

$$\{1, x, \dots, x^n, y, xy, x^2y, \dots, x^ny\}, \text{ where } \deg(y) = 2n.$$

Applying the map r on both sides of (8.4) for $i = 0$, we get the relation

$$x^{n+1} = 2xy.$$

We still need to find the relation between y^2 and the top generator x^ny . For this, we use Theorem 2.4 to integrate β_0^2 on M :

$$\begin{aligned} \int_M \beta_0^2 &= \int_M y^2 = \int_X \frac{\beta_0^2|_X}{e^{S^1}(N_X)} + \int_Y \frac{\beta_0^2|_Y}{e^{S^1}(N_Y)} = 0 + \int_Y e^{S^1}(N_Y) \\ &= (-1)^n \int_Y \frac{(t-v)^{n+1}}{t-2v} = (-1)^n \frac{1}{t} \int_Y (t-v)^{n+1} \left(1 + \frac{2v}{t} + \dots + \left(\frac{2v}{t}\right)^n\right) \end{aligned}$$

$$\begin{aligned}
&= (-1)^n \frac{1}{t} \int_Y \left(t^{n+1} - \binom{n+1}{1} t^n v + \cdots + \binom{n+1}{n} (-1)^n t v^n \right) \left(1 + \frac{2v}{t} + \cdots + \left(\frac{2v}{t} \right)^n \right) \\
&= \frac{1 + (-1)^n}{2}.
\end{aligned}$$

Hence

$$H^*(M; \mathbb{Z}) = \begin{cases} \mathbb{Z}[x, y]/(x^{n+1} - 2xy, y^2), & \text{if } n \text{ is odd,} \\ \mathbb{Z}[x, y]/(x^{n+1} - 2xy, y^2 - x^n y), & \text{if } n \text{ is even.} \end{cases}$$

□

Now we can complete the proof of Theorem 1.6:

Proof of Theorem 1.6. (1) and (2) follow from Proposition 5.1, (3) follow from Lemma 6.1, (4) and (5) follow from Proposition 8.1, and (6) follows from Lemma 7.1 and Proposition 7.3. □

9. THE RING $H^*(M; \mathbb{Z})$ DETERMINES THE OTHER DATA — PROOF OF THEOREM 1.8

In this section, we prove Theorem 1.8.

Proposition 9.1. *Let (M, ω) be a compact symplectic manifold admitting a Hamiltonian S^1 action such that M^{S^1} consists of two connected components X and Y with $\dim(X) + \dim(Y) = \dim(M)$. If $H^*(M; \mathbb{Z}) \cong H^*(\tilde{G}_2(\mathbb{R}^{2n+2}); \mathbb{Z})$ as rings, then $H^*(X; \mathbb{Z}) \cong H^*(\mathbb{CP}^n; \mathbb{Z})$ and $H^*(Y; \mathbb{Z}) \cong H^*(\mathbb{CP}^n; \mathbb{Z})$ as rings, with the natural induced symplectic orientations on X and Y .*

Proof. Since $H^*(M; \mathbb{Z}) \cong H^*(\tilde{G}_2(\mathbb{R}^{2n+2}); \mathbb{Z})$ (as groups), and M is a compact oriented manifold, we have $\dim(M) = 4n$, and $b_{2i}(M) = 1$ if $0 \leq 2i \leq 4n$ and $2i \neq 2n$, and $b_{2n}(M) = 2$.

Let ϕ be the moment map and assume $\phi(X) < \phi(Y)$. Assume $\dim(X) > \dim(Y)$, then $\dim(X) > 2n$, so $b_{\dim(X)}(M) = 1$ by the first paragraph. But, by (3.3) and the fact $\text{codim}(Y) = \dim(X)$, we have $b_{\dim(X)}(M) = b_{\dim(X)}(X) + 1 = 2$, a contradiction. So $\dim(X) \leq \dim(Y)$. Similarly, argue using $-\phi$, we have $\dim(Y) \leq \dim(X)$. Hence $\dim(X) = \dim(Y) = 2n$.

Using ϕ as a Morse-Bott function, since $\text{codim}(Y) = \dim(X) = 2n$, the natural restriction map

$$H^i(M; \mathbb{Z}) \rightarrow H^i(X; \mathbb{Z})$$

is an isomorphism for all $0 \leq i < 2n$. If $[\omega]$ is primitive integral, then by assumption, $H^*(M; \mathbb{Z}) \cong H^*(\tilde{G}_2(\mathbb{R}^{2n+2}); \mathbb{Z})$ has a subring $\mathbb{Z}[[\omega]]/[\omega]^{n+1}$ in degrees less than $2n + 1$. By the above isomorphisms, if $u = [\omega|_X]$, then the degree less than $2n$ terms of $H^*(X; \mathbb{Z})$ has the ring structure $\mathbb{Z}[u]/u^n$. If u^n is not a generator of $H^{2n}(X; \mathbb{Z})$, then au^n , where $0 < a < 1$, is a generator. Since $u \in H^2(X; \mathbb{Z})$ is a generator, by Poincaré duality, $au^{n-1} \in H^{2n-2}(X; \mathbb{Z})$ is a generator, a contradiction. Hence $H^*(X; \mathbb{Z}) = \mathbb{Z}[u]/u^{n+1}$.

Similarly, using $-\phi$, we get that Y has the integral cohomology ring of \mathbb{CP}^n . \square

Now we can prove Theorem 1.8.

Proof of Theorem 1.8. We may assume that $[\omega]$ is a primitive integral class. Let $u = [\omega|_X]$ and $v = [\omega|_Y]$. By Proposition 9.1, the assumption implies that

$$H^*(X; \mathbb{Z}) = \mathbb{Z}[u]/u^{n+1} \quad \text{and} \quad H^*(Y; \mathbb{Z}) = \mathbb{Z}[v]/v^{n+1}.$$

If $\dim(M) = 4n = 4$, then X and Y are diffeomorphic to S^2 , so M is diffeomorphic to an S^2 -bundle over S^2 . Since $H^*(M; \mathbb{Z}) \cong H^*(\tilde{G}_2(\mathbb{R}^4); \mathbb{Z})$, M is diffeomorphic to $S^2 \times S^2$. Since $\text{codim}(X) = \text{codim}(Y) = 2$ and the action is effective, the action must be semifree. Other claims naturally follow.

For $\dim(M) > 4$, the claims follow from Theorem 1.6. \square

10. WHEN THE MANIFOLD IS KÄHLER — PROOF OF THEOREM 1.9

The proof of Theorem 1.9 uses Theorems 1.5, 1.6, 1.8 and the following result which is part of [9, Prop. 4.2].

Proposition 10.1. *Let (M, ω, J) be a compact Kähler manifold of complex dimension n , which admits a holomorphic Hamiltonian circle action. Assume that $[\omega]$ is an integral class. If $c_1(M) = n[\omega]$, then M is S^1 -equivariantly biholomorphic to $\tilde{G}_2(\mathbb{R}^{n+2})$, equipped with a standard circle action.*

By Theorems 1.5, 1.6, and 1.8, any of the conditions in Theorem 1.9 implies that $c_1(M) = 2n[\omega]$ for a suitable integral Kähler class $[\omega]$. Then by Proposition 10.1, M is S^1 -equivariantly biholomorphic to $\tilde{G}_2(\mathbb{R}^{2n+2})$, equipped with a standard circle action.

Now, let

$$f: (M, \omega, J) \rightarrow (\tilde{G}_2(\mathbb{R}^{2n+2}), \omega', J')$$

be the S^1 -equivariant biholomorphism. By rescaling the symplectic form (in high dimension) or by changing the symplectic class (in dimension 4), we may assume that ω and $f^*\omega'$ represent the same cohomology class on M . Then the family of forms $\omega_t = (1-t)\omega + tf^*\omega'$ on M , where $t \in [0, 1]$, represent the same cohomology class. Now we show that each ω_t is nondegenerate. For any point $x \in M$, suppose $X \in T_x M$ is such that $\omega_t(X, Y) = 0$ for all $Y \in T_x M$. In particular, if $Y = JX$, then $\omega_t(X, JX) = 0$. Using the facts $\omega(X, JX) \geq 0$, $f_*(JX) = J'f_*X$, and $\omega'(f_*X, J'f_*X) \geq 0$, we get $X = 0$. By Moser's method [13], there exists an S^1 -equivariant isotopy Φ_t such that $\Phi_t^*\omega_t = \omega$, in particular, $\Phi_1^*f^*\omega' = \omega$. So $f \circ \Phi_1$ is the S^1 -equivariant symplectomorphism we are looking for.

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